

Wave Motion

e-content for B.Sc Physics (Honours)

B.Sc Part-I

Paper-I

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Wave motion

Type of waves:

There are two types of wave motion.

(i) Transverse wave: When the particles of the medium vibrate at right angles to the direction of propagation of the wave, the wave is said to be a transverse wave.

(ii) Longitudinal wave: When the particles of the medium vibrates parallel to the direction of propagation of the wave the wave is called a longitudinal wave. e.g. sound wave in solids, liquids and gases.

Progressive wave: A wave propagating from one point to another in a medium without being subjected to any boundary condition, is called a progressive wave.

Q. Write down the characteristics of progressive wave

(i) Every particle describes simple harmonic motion along the direction of propagation of wave, there being a change of phase from point to point.

(ii) The wave velocity in a given medium is a constant determined by the density and the elastic constant of the medium.

(iii) Only the energy is carried by the advancement of the waveform in the direction of propagation of the wave.

(iv) The phase difference between two vibrating particles on the line of propagation is proportional to the path difference between the particles.

Q. Write down the equation of the plane progressive wave.

Consider a wave moves along the positive direction of x with a velocity v . Let the displacement at any instant of time t at $x = 0$ is

$$y = a \sin \omega t$$

Here, v be the wave velocity. We have from fig. for λ displacement phase change is 2π

So for x displacement the phase change $\phi = \frac{2\pi}{\lambda}x$

So we get motion at P is

$$y = a \sin(\omega t - \phi)$$

$$y = a \sin\left(\omega t - \frac{2\pi}{\lambda}x\right) = f(vt - x)$$

$$y = a \sin \frac{2\pi}{\lambda} (vt - x)$$

where $\omega = 2\pi n = \frac{2\pi v}{\lambda}$.

If the wave moves towards the negative direction of the x-axis, the displacement

$$y = a \sin \frac{2\pi}{\lambda} (vt + x) = f(vt + x)$$

Q. Define Phase velocity

At any instant in a progressive wave, the quantity $vt - x$ and hence $f(vt - x)$ remains the same at all points on a plane perpendicular to the x-axis. Thus the wave fronts are plane., so that $f(vt - x)$ represents a plane wave propagating in the positive x-direction. The quantity $vt - x$ is the phase.

Let $y(x, t)$ denotes the value of the field parameter at x at time t . As the wave moves in the positive x-direction, the same value occurs at $x + dx$ at time $t + dt$. Hence

$$y(x, t) = y(x + dx, t + dt) = \text{constant}$$

$$f(vt - x) = f(v(t + dt) - (x + dx)) = \text{constant}$$

So, we get

$$vt - x = \text{constant}$$

$$vdt - dx = 0$$

$$v = \frac{dx}{dt} = v_p$$

Hence, wave velocity (v) = (v_p) Phase velocity

OR

At any instant in a progressive wave, the quantity $\omega t - kx$ and hence $f(\omega t - kx)$ remains the same at all points on a plane perpendicular to the x-axis. Thus the wave fronts are plane., so that $f(\omega t - kx)$ represents a plane wave propagating in the positive x-direction. The quantity $\omega t - kx$ is the phase.

Let $y(x, t)$ denotes the value of the field parameter at x at time t . As the wave moves in the positive x-direction, the same value occurs at $x + dx$ at time $t + dt$. Hence

$$y(x, t) = y(x + dx, t + dt) = \text{constant}$$

$$f(\omega t - kx) = f(\omega(t + dt) - k(x + dx)) = \text{constant}$$

So, we get

$$\omega t - kx = \text{constant}$$

$$\omega dt - kdx = 0$$

$$v = \frac{dx}{dt} = \frac{\omega}{k} = v_p$$

Hence, wave velocity (v) = (v_p) Phase velocity

Q. Derive the differential equation of wave equation in one dimension.

Let a plane progressive wave propagating in the +ve x-direction. The wave form is represented by

$$y = f(vt - x)$$

Let $z = vt - x$, so we get

$$\frac{\partial z}{\partial t} = v$$

$$\frac{\partial z}{\partial x} = -1$$

Now,

$$\begin{aligned} \frac{\partial y}{\partial x} &= \frac{\partial y}{\partial z} \frac{\partial z}{\partial x} = -\frac{\partial y}{\partial z} \\ \frac{\partial^2 y}{\partial x^2} &= \frac{\partial}{\partial x} \frac{\partial y}{\partial x} = -\frac{\partial}{\partial z} \frac{\partial y}{\partial z} \frac{\partial z}{\partial x} = -1 \cdot -1 \cdot \frac{\partial^2 y}{\partial z^2} \end{aligned} \quad (1)$$

and

$$\frac{\partial y}{\partial t} = \frac{\partial y}{\partial z} \frac{\partial z}{\partial t} = v \frac{\partial y}{\partial z}$$

$$\begin{aligned} \frac{\partial^2 y}{\partial t^2} &= v \frac{\partial}{\partial t} \frac{\partial y}{\partial z} = v \frac{\partial}{\partial z} \frac{\partial y}{\partial z} \frac{\partial z}{\partial t} = v^2 \frac{\partial^2 y}{\partial z^2} \\ \frac{\partial^2 y}{\partial t^2} &= v^2 \frac{\partial^2 y}{\partial z^2} \end{aligned} \quad (2)$$

From (1) and (2), we get

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$$

Here v is the phase or wave velocity of the wave. This equation is known as the differential equation in one dimension for plane waves. The general solution is

$$y = f_1(vt - x) + f_2(vt + x)$$

Note: In three dimension, the differential equation for waves takes the form

$$\frac{\partial^2 y}{\partial t^2} = v^2 \nabla^2 y$$

Where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$Y = a \sin(\omega t - k_1 x - k_2 y - k_3 z)$$

$$Y = a \sin(\omega t - (k_1 \hat{i} + k_2 \hat{j} + k_3 \hat{k}) \cdot (x \hat{i} + y \hat{j} + z \hat{k}))$$

$$Y = a \sin(\omega t - \vec{k} \cdot \vec{r})$$

where $\vec{k} = k_1\hat{i} + k_2\hat{j} + k_3\hat{k}$.

Q. Derive the relation between particle velocity (U) and wave velocity (v).

When a progressive wave travels through a medium, the displacement of a particle of the medium at any instant of time

$$y = a \sin \frac{2\pi}{\lambda} (vt - x)$$

So, the velocity of the particle

$$U = \frac{dy}{dt} = \frac{2\pi v}{\lambda} a \cos \frac{2\pi}{\lambda} (vt - x) \quad (1)$$

Again we have

$$\frac{dy}{dx} = -\frac{2\pi}{\lambda} a \cos \frac{2\pi}{\lambda} (vt - x) \quad (2)$$

So, we get from (1) and (2)

$$U = -v \frac{dy}{dx}$$

Note: For longitudinal wave $\frac{dy}{dx}$ represents the rarefaction or contraction.

Q. Calculate the energy of a progressive wave.

When a progressive wave travels through a medium, the displacement of a particle of the medium at any instant of time

$$y = a \sin \frac{2\pi}{\lambda} (vt - x)$$

So, the velocity of the particle

$$U = \frac{dy}{dt} = \frac{2\pi v}{\lambda} a \cos \frac{2\pi}{\lambda} (vt - x)$$

So, acceleration of the particle

$$f = \frac{d^2y}{dt^2} = -\frac{4\pi^2 v^2}{\lambda^2} a \sin \frac{2\pi}{\lambda} (vt - x)$$

$$f = \frac{d^2y}{dt^2} = -\frac{4\pi^2v^2}{\lambda^2}y$$

Let ρ be the density of the medium. So, kinetic energy per unit volume at any instant of time

$$\begin{aligned} E_{K.E.} &= \frac{1}{2}\rho\left(\frac{dy}{dt}\right)^2 \\ E_{K.E.} &= \frac{1}{2}\rho\left(\frac{2\pi v}{\lambda}a\cos\frac{2\pi}{\lambda}(vt-x)\right)^2 \\ E_{K.E.} &= \frac{1}{2}\rho\frac{4\pi^2v^2}{\lambda^2}a^2\cos^2\frac{2\pi}{\lambda}(vt-x) \end{aligned} \quad (1)$$

Now, potential energy $dE_{P.E.}$ = work done for the displacement $dy = dy \times force$

$$\begin{aligned} dE_{P.E.} &= dy \times \rho\frac{d^2y}{dt^2} \\ dE_{P.E.} &= \rho\frac{4\pi^2v^2}{\lambda^2}ydy \end{aligned}$$

Total potential energy for the displacement y

$$\begin{aligned} E_{P.E.} &= \rho\frac{4\pi^2v^2}{\lambda^2}\int_0^y ydy \\ E_{P.E.} &= \rho\frac{4\pi^2v^2}{\lambda^2}\frac{y^2}{2} \\ E_{P.E.} &= \frac{1}{2}\rho\frac{4\pi^2v^2}{\lambda^2}a^2\sin^2\frac{2\pi}{\lambda}(vt-x) \end{aligned} \quad (2)$$

Total energy per volume at any instant of time

$$\begin{aligned} E &= E_{K.E.} + E_{P.E.} \\ E &= \frac{1}{2}\rho\frac{4\pi^2v^2}{\lambda^2}a^2\cos^2\frac{2\pi}{\lambda}(vt-x) + \frac{1}{2}\rho\frac{4\pi^2v^2}{\lambda^2}a^2\sin^2\frac{2\pi}{\lambda}(vt-x) \\ E &= \frac{1}{2}\rho\frac{4\pi^2v^2}{\lambda^2}a^2\left(\cos^2\frac{2\pi}{\lambda}(vt-x) + \sin^2\frac{2\pi}{\lambda}(vt-x)\right) \\ E &= \frac{1}{2}\rho\frac{4\pi^2v^2}{\lambda^2}a^2 = constant \end{aligned}$$

Q. What is the distribution of energy in a plane progressive wave.

We have the kinetic energy per unit volume at any instant of time

$$E_{K.E.} = \frac{1}{2} \rho \frac{4\pi^2 v^2}{\lambda^2} a^2 \cos^2 \frac{2\pi}{\lambda} (vt - x) \quad (1)$$

So, mean kinetic energy for a full wave length

$$\begin{aligned} \bar{E} &= \frac{1}{\lambda} \int_0^\lambda \frac{1}{2} \rho \frac{4\pi^2 v^2}{\lambda^2} a^2 \cos^2 \frac{2\pi}{\lambda} (vt - x) d\lambda \\ \bar{E} &= \frac{1}{\lambda} \frac{1}{2} \rho \frac{4\pi^2 v^2}{\lambda^2} a^2 \int_0^\lambda \cos^2 \frac{2\pi}{\lambda} (vt - x) d\lambda \\ \bar{E} &= \frac{1}{\lambda} \frac{1}{2} \rho \frac{4\pi^2 v^2}{\lambda^2} a^2 \int_0^\lambda \left(1 + \cos \frac{4\pi}{\lambda} (vt - x) \right) d\lambda \\ \bar{E} &= \frac{1}{\lambda} \frac{1}{2} \rho \frac{4\pi^2 v^2}{\lambda^2} a^2 \frac{\lambda}{2} \\ \bar{E} &= \frac{1}{2} \times E \end{aligned}$$

Q. Calculate the distribution of pressure in longitudinal waves.

Let a longitudinal wave propagates along x-axis in a medium (fluid). Now, we consider a layer AB at a distance x from O and thickness of the layer AB = dx.

Let α is the area of the layer.

So, volume of the layer $V = \alpha dx$

Let dp be the pressure difference between the two faces.

So the particles on the planes A and B are displaced due to the excess pressure dP produced by the progressive wave. Let displacement of the layer A is y and that of B is $y + dy$ According to the fig.

$$OA' = x + y$$

$$OB' = x + dx + y + dy = x + dx + y + \frac{\partial y}{\partial x} dx$$

Thickness of the displaced layer

$$A'B' = OB' - OA' = (x + dx + y + \frac{\partial y}{\partial x} \delta x) - (x + y) = dx + \frac{\partial y}{\partial x} \delta x$$

Volume of the displaced layer

$$V' = \alpha(dx + \frac{\partial y}{\partial x} \delta x)$$

Change of volume

$$dV = V' - V = \alpha(dx + \frac{\partial y}{\partial x} \delta x) - \alpha dx = \alpha \frac{\partial y}{\partial x} \delta x$$

We have from the definition of Bulk modulus

$$K = -\frac{dp}{\frac{dV}{V}} = -\frac{dP}{\frac{\alpha \frac{\partial y}{\partial x} \delta x}{\alpha dx}}$$

Excess pressure on the layer of the medium (fluid)

$$dP = -K \frac{\partial y}{\partial x}$$

This pressure is known as the sound pressure or acoustic pressure. When a progressive wave travels through a medium, the displacement of a particle of the medium at any instant of time

$$y = a \sin \frac{2\pi}{\lambda} (vt - x)$$

Hence,

$$\frac{\partial y}{\partial x} = -\frac{2\pi}{\lambda} a \cos \frac{2\pi}{\lambda} (vt - x)$$

So, we get pressure difference

$$dp = -K \frac{2\pi}{\lambda} a \cos \frac{2\pi}{\lambda} (vt - x)$$

$$dp = K \frac{2\pi}{\lambda} a \sin \frac{2\pi}{\lambda} \left(vt - x + \frac{\pi}{2} \right)$$

So, we see that dp lag before y by $\pi/2$.

Q. State Principle of superposition.

When two or more waves propagating independently of one another in a medium at the same time superimpose, the resultant displacement, velocity, and acceleration of any particle of the medium in the region of overlap is the vector sum of the displacements, velocities and accelerations of the particle caused by the individual waves. This is known as the principle of superposition.

Let $\vec{y}_1, \vec{y}_2, \vec{y}_3, \dots$ be the displacement. According to the principle of superposition

$$\vec{y} = \vec{y}_1 + \vec{y}_2 + \vec{y}_3 + \dots$$

Q. Define group velocity.

If two or more plane simple harmonic waves of the same amplitude but different frequencies superimpose, a group of waves is formed. The amplitude of the group changes with distance, and the velocity with which the maximum of the wave group travels is referred to as the group velocity. The energy is transmitted with the group velocity.

We consider two waves of equal amplitude A and slightly different angular frequencies ω and $\omega + d\omega$, traveling with the propagation constant k and $k + dk$. i.e. the displacement

$$y_1 = A \sin(\omega t - kx)$$

and

$$y_2 = A \sin((\omega + d\omega)t - (k + dk)x)$$

So, the resultant displacement

$$y = y_1 + y_2 = A \sin(\omega t - kx) + A \sin((\omega + d\omega)t - (k + dk)x)$$

$$y = 2A \cos\left(\frac{td\omega - xdk}{2}\right) \sin\left(\left(\omega + \frac{d\omega}{2}\right)t - \left(k + \frac{dk}{2}\right)x\right)$$

The phase velocity of the composite wave is

$$v_p = \frac{\omega + d\omega/2}{k + dk/2} \neq \frac{\omega}{k}$$

The amplitude represented by the cosine term advance with the group velocity v_g . If the phase $\frac{td\omega - xdk}{2}$ associated with it progresses to $(x + dx)$ at time $(t + dt)$, we have

$$td\omega - xdk = (t + dt)d\omega - (x + dx)dk$$

So, the group velocity

$$v_g = \frac{dx}{dt} = \frac{d\omega}{dk}$$

Again, we have

$$\omega = vk$$

$$d\omega = vdk + kdv$$

$$v_g = v + k \frac{dv}{dk}$$

Again,

$$k = \frac{2\pi}{\lambda}$$

$$\lambda = \frac{2\pi}{k}$$

$$\frac{d\lambda}{dk} = -\frac{2\pi}{k^2}$$

So, we get

$$v_g = v + \frac{dv}{d\lambda} \frac{d\lambda}{dk}$$

$$v_g = v - \frac{2\pi}{k^2} k \frac{dv}{dk}$$

$$v_g = v - \frac{2\pi}{k} \frac{dv}{d\lambda}$$

This gives the relation between group velocity and the phase velocity.

For a non-dispersive medium, $\frac{dv}{d\lambda} = 0$. Hence

$$v_g = v = v_p$$

i.e. The group velocity and the phase velocity are equal.

In a dispersive medium v increases with increasing λ , Hence, $\frac{dv}{d\lambda} = +ve$ So, $v_g < v$
i.e. the group velocity is less than the phase velocity.

Q. What is stationary wave.

When two identical progressive acoustic waves moving in a medium along the same straight line with the same velocity in opposite directions superimpose produce the stationary waves or standing waves.

This waves are confined to the region of the medium where the progressive waves overlap. They do not advance through the medium, but alternately expand and shrink.

Q. Explain the formation of stationary waves by analytical method.

We consider the particle displacement for the wave propagating in the +ve x-direction

$$y_1 = a \sin \frac{2\pi}{\lambda} (vt - x)$$

The wave propagating in the -ve x-direction

$$y_2 = a \sin \frac{2\pi}{\lambda} (vt + x)$$

When these two waves superpose, the resultant particle displacement is

$$\begin{aligned} y &= y_1 + y_2 \\ y &= a \sin \frac{2\pi}{\lambda} (vt - x) + a \sin \frac{2\pi}{\lambda} (vt + x) \\ y &= 2a \cos \frac{2\pi x}{\lambda} \sin \frac{2\pi vt}{\lambda} \\ y &= A \sin \frac{2\pi vt}{\lambda} \end{aligned} \tag{1}$$

where amplitude

$$A = 2a \cos \frac{2\pi x}{\lambda}$$

The equation (1) is known as stationary wave. This equation shows that the amplitude A of the stationary wave is not a constant, it is a periodic function of x .

Position of Nodes:

At nodes, $A = 0$, Hence,

$$\cos \frac{2\pi x}{\lambda} = 0 = \cos(2n + 1) \frac{\pi}{2}$$

where, $n = 0, \pm 1, \pm 2, \text{etc.}$

$$\frac{2\pi x}{\lambda} = (2n + 1) \frac{\pi}{2}$$

$$x = x_n = (2n + 1) \frac{\lambda}{4}$$

So, for $n = 0$,

$$x_0 = \frac{\lambda}{4}$$

So, for $n = 1$,

$$x_1 = 3 \frac{\lambda}{4}$$

$$x_1 - x_0 = 3 \frac{\lambda}{4} - \frac{\lambda}{4} = \frac{\lambda}{2}$$

In general

$$x_{n+1} - x_n = (2(n + 1) + 1) \frac{\lambda}{4} - (2n + 1) \frac{\lambda}{4} = \frac{\lambda}{2}$$

Hence, the distance between two consecutive nodes is $\frac{\lambda}{2}$

Position of antinodes:

At antinodes, $A = \text{max.} = \pm 2a$, Hence,

$$\cos \frac{2\pi x}{\lambda} = 1 = \cos n\pi$$

where, $n = 0, \pm 1, \pm 2, \text{etc.}$

$$\frac{2\pi x}{\lambda} = n\pi$$

$$x = x_n = \frac{n\lambda}{2}$$

So, for $n = 0$,

$$x_0 = 0$$

So, for $n = 1$,

$$x_1 = \frac{\lambda}{2}$$

$$x_1 - x_1 = \frac{\lambda}{2} - 0 = \frac{\lambda}{2}$$

In general

$$x_{n+1} - x_n = (n+1)\frac{\lambda}{2} - n\frac{\lambda}{2} = \frac{\lambda}{2}$$

Hence, the distance between two consecutive antinodes is $\frac{\lambda}{2}$.

Q. Calculate the particle velocity and acceleration in a stationary wave.

We have the equation of a stationary wave

$$y = 2a \cos \frac{2\pi x}{\lambda} \sin \frac{2\pi vt}{\lambda}$$

So, the particle velocity

$$U = \frac{dy}{dt} = \frac{4\pi av}{\lambda} \cos \frac{2\pi x}{\lambda} \cos \frac{2\pi vt}{\lambda}$$

Position of zero velocity:

At nodes, $U = 0$, Hence,

$$\cos \frac{2\pi x}{\lambda} = 0 = \cos(2n+1)\frac{\pi}{2}$$

where, $n = 0, \pm 1, \pm 2, \text{etc.}$

$$\frac{2\pi x}{\lambda} = (2n+1)\frac{\pi}{2}$$

$$x = x_n = (2n+1)\frac{\lambda}{4}$$

Position of maximum velocity:

At antinodes, $U = \text{max.} = \frac{4\pi av}{\lambda}$, Hence,

$$\cos \frac{2\pi x}{\lambda} = 1 = \cos n\pi$$

where, $n = 0, \pm 1, \pm 2, \text{etc.}$

$$\frac{2\pi x}{\lambda} = n\pi$$
$$x = x_n = \frac{n\lambda}{2}$$

Acceleration:

Acceleration of the particle

$$f = \frac{dU}{dt} = -\frac{8\pi^2 av^2}{\lambda^2} \cos \frac{2\pi x}{\lambda} \sin \frac{2\pi vt}{\lambda}$$
$$f = -\frac{8\pi^2 v^2}{\lambda^2} y$$

Q. Calculate the variation of pressure at node and antinodes in a stationary wave.

We have the equation of a stationary wave

$$y = 2a \cos \frac{2\pi x}{\lambda} \sin \frac{2\pi vt}{\lambda}$$

We have the variation of excess pressure

$$\Delta P = -K \frac{dy}{dx} = \frac{4\pi av}{\lambda} \sin \frac{2\pi x}{\lambda} \sin \frac{2\pi vt}{\lambda}$$

At the Position of nodes:

At nodes, $\Delta P = \text{max} = \frac{4\pi av}{\lambda}$, Hence,

$$\sin \frac{2\pi x}{\lambda} = 1 = \sin(2n + 1) \frac{\pi}{2}$$

where, $n = 0, \pm 1, \pm 2, \text{etc.}$

$$\frac{2\pi x}{\lambda} = (2n + 1) \frac{\pi}{2}$$

$$x = x_n = (2n + 1) \frac{\lambda}{4}$$

At the Position of antinodes:

At antinodes, $\Delta P = 0$, Hence,

$$\sin \frac{2\pi x}{\lambda} = 0 = \sin n\pi$$

where, $n = 0, \pm 1, \pm 2, \text{etc.}$

$$\frac{2\pi x}{\lambda} = n\pi$$

$$x = x_n = \frac{n\lambda}{2}$$

Q. Explain the stationary wave produced by reflection.

If a boundary surface is placed in the path of a progressive wave, the wave is reflected from the surface. The reflected wave travels backwards and superimpose on the forward wave, thus producing stationary waves.

Let the particle displacement for the incident wave moving in the positive x-direction be

$$y_1 = a \sin \frac{2\pi}{\lambda} (vt - x)$$

The reflected wave moves in the negative x-direction

$$y_2 = a_r \sin \frac{2\pi}{\lambda} (vt + x) = R a \sin \frac{2\pi}{\lambda} (vt + x)$$

where $R (< 1)$ is the reflection coefficient. This is defined as the ratio of the reflected amplitude (a_r) to the incident amplitude (a). i.e.

$$R = \frac{a_r}{a}$$

Let the reflector is placed at $x = 0$. So we get

$$y_1 = a \sin \frac{2\pi}{\lambda} vt$$

and

$$y_2 = R a \sin \frac{2\pi}{\lambda} vt$$

At this point, the resultant displacement of the particle

$$y = y_1 + y_2 = (1 + R) a \sin \frac{2\pi}{\lambda} vt$$

Pressure calculation: We have the excess pressure

$$P = -K \frac{dy}{dx}$$

So, excess pressure due to incident wave

$$P_{in} = -K \left(\frac{dy}{dx} \right)_{x=0} = \frac{2\pi K a}{\lambda} \cos \frac{2\pi}{\lambda} vt$$

So, excess pressure due to reflected wave

$$P_{re} = -K \left(\frac{dy}{dx} \right)_{x=0} = -\frac{2\pi K a}{\lambda} R \cos \frac{2\pi}{\lambda} vt$$

Hence, the resultant excess pressure at $x = 0$

$$P = P_{in} + P_{re} = \frac{2\pi K a}{\lambda} (1 - R) \cos \frac{2\pi}{\lambda} vt$$

Case I: If the reflector is perfectly rigid, then the particle displacement $y = 0$ at $x = 0$. Hence

$$R = -1 = e^{i\pi}$$

This shows that at the point of reflection, there is a phase shift of $\phi = \pi$ between the incident and reflected wave. Here, the boundary is a displacement node and a pressure antinode. .

Case II: If the reflector move freely, the resultant excess pressure $P = 0$ at $x = 0$. Hence

$$R = 1 = e^{i.0}$$

So, there is no change in phase($\phi = 0$). Here the boundary is a pressure node and a displacement antinode.

The above two cases represent extreme situations. If the reflecting wall is not perfectly rigid, R lies between -1 and $\phi = 0$. The reflected wave has a smaller amplitude and carries less energy than incident wave.

Q. What is interference of sound.

When two progressive acoustic waves of same amplitude and but different phases moving in a medium along the same direction with the same velocity superimpose at a point produce the interference.

Q. Find out the conditions for interference of sound

We consider two progressive waves of same amplitude and wavelength as

$$y_1 = a \sin \frac{2\pi}{\lambda} (vt - x_1)$$

and

$$y_2 = a \sin \frac{2\pi}{\lambda} (vt - x_2)$$

According to the Principle of superposition, the resultant displacement

$$y = y_1 + y_2 = a \sin \frac{2\pi}{\lambda} (vt - x_1) + a \sin \frac{2\pi}{\lambda} (vt - x_2)$$

$$y = 2a \cos \frac{\pi}{\lambda} (x_2 - x_1) \sin \frac{2\pi}{\lambda} \left(vt - \frac{x_1 + x_2}{2} \right)$$

$$y = A \sin \frac{2\pi}{\lambda} \left(vt - \frac{x_1 + x_2}{2} \right)$$

This is a S.H. Motion of amplitude

$$A = 2a \cos \frac{\pi}{\lambda} (x_2 - x_1)$$

which depends upon the path difference $x_2 - x_1$.

Case I: The amplitude A will be minimum i.e. $A = 0$, when

$$\cos \frac{\pi}{\lambda} (x_2 - x_1) = \cos(2n + 1) \frac{\pi}{2}$$

$$\frac{\pi}{\lambda}(x_2 - x_1) = (2n + 1)\frac{\pi}{2}$$

Hence, path difference for minimum sound

$$x_2 - x_1 = (2n + 1)\frac{\lambda}{2}$$

So, minimum sound is obtained, when two waves coincide at a point in opposite phases.

Case II: The amplitude A will be maximum, i.e. $A = 2a$, when

$$\cos \frac{\pi}{\lambda}(x_2 - x_1) = \cos n\pi$$

$$\frac{\pi}{\lambda}(x_2 - x_1) = n\pi = (2n)\frac{\pi}{2}$$

Hence, path difference for maximum sound

$$x_2 - x_1 = 2n\frac{\lambda}{2}$$

So, maximum sound is obtained when two waves coincide at point in same phase.

Condition for interference:

- (i) For interference, amplitude (a) and wavelength (λ) must be same.
- (ii) For interference, two waves must propagate along the same direction.
- (iii) For interference, path difference for minimum sound

$$x_2 - x_1 = (2n + 1)\frac{\lambda}{2}$$

So, minimum sound is obtained, when two waves coincide at a point in opposite phases.

- (iv) For interference, path difference for maximum sound

$$x_2 - x_1 = 2n\frac{\lambda}{2}$$

So, maximum sound is obtained when two waves coincide at point in same phase.

Q. Define beats

When two simple harmonic motions of slightly different frequencies superimpose, the amplitude of the resultant vibration changes regularly with time between a maximum and a minimum. This phenomenon is referred to as beats.

This is observed when two tuning forks or two sources of sound of nearly equal frequencies are sounded together. The method of beats is a very important one in the measurement of an unknown frequency.

Q. Give the analytical treatment of beats or What happens when two vibrations of slightly different frequencies along same straight line.

Let two S.H. M. are

$$y_1 = a \sin 2\pi n_1 t$$

and

$$y_2 = a \sin 2\pi n_2 t$$

Here n_1 is slightly greater than n_2 . Due to superposition, the resultant displacement is

$$\begin{aligned} y &= y_1 + y_2 = a \sin 2\pi n_1 t + a \sin 2\pi n_2 t \\ y &= 2a \cos 2\pi \left(\frac{n_1 - n_2}{2} \right) t \sin 2\pi \left(\frac{n_1 + n_2}{2} \right) t \\ y &= A \sin 2\pi \left(\frac{n_1 + n_2}{2} \right) t \end{aligned}$$

where the amplitude $A = 2a \cos 2\pi \left(\frac{n_1 - n_2}{2} \right) t$ changes with time.

Beat frequency: Number of maximum sound or minimum sound is known as beat frequency. Hence

Beat frequency = difference of frequency = $n_1 - n_2$

Case I: For maximum sound, the amplitude $A = 2a$, when

$$\cos 2\pi \left(\frac{n_1 - n_2}{2} \right) t = \pm 1 = \cos m\pi$$

where $m = 0, \pm 1 \pm 2, \dots$

$$2\pi\left(\frac{n_1 - n_2}{2}\right)t = m\pi$$

$$t = t_m = \frac{m}{n_1 - n_2}$$

So, the time interval between two maximum sound

$$t_{m+1} - t_m = \frac{m+1}{n_1 - n_2} - \frac{m}{n_1 - n_2} = \frac{1}{n_1 - n_2}$$

Case II: For minimum sound, the amplitude $A = 0$, when

$$\cos 2\pi\left(\frac{n_1 - n_2}{2}\right)t = 0 = \cos(2m+1)\frac{\pi}{2}$$

where $m = 0, 1, 2, \dots$

$$2\pi\left(\frac{n_1 - n_2}{2}\right)t = (2m+1)\frac{\pi}{2}$$

$$t = t_m = \frac{(2m+1)}{2(n_1 - n_2)}$$

So, the time interval between two minimum sound

$$t_{m+1} - t_m = \frac{2m+3}{2(n_1 - n_2)} - \frac{2m+1}{2(n_1 - n_2)} = \frac{1}{n_1 - n_2}$$

Q. Calculate the velocity of propagation of plane longitudinal waves in a elastic fluid.

For this calculation we make the following assumption.

- (i) The medium is homogeneous and isotropic.
- (ii) Dissipative forces originating from viscosity and thermal conduction are absent.
- (iii) The effect of gravity is negligible. Hence, the pressure and the density are the same everywhere in the medium.
- (iv) Hook's law holds good.

Let a longitudinal wave propagates along x-axis in a medium (fluid). Now, we consider a layer AB at a distance x from O and thickness of the layer AB= dx.

Let α is the area of the layer.

So, volume of the layer $V = \alpha dx$

Let there is an excess pressure between the faces of the layer AB. So the particles on the planes A and B are displaced due to the excess pressure ΔP produced by the progressive wave. Let displacement of the layer A is y and that of B is $y + dy$ According to the fig.

$$OA' = x + y$$

$$OB' = x + dx + y + dy = x + dx + y + \frac{\partial y}{\partial x} \delta x$$

Thickness of the displaced layer

$$A'B' = OB' - OA' = (x + dx + y + \frac{\partial y}{\partial x} \delta x) - (x + y) = dx + \frac{\partial y}{\partial x} \delta x$$

Volume of the displaced layer

$$V' = \alpha(dx + \frac{\partial y}{\partial x} \delta x)$$

Change of volume

$$dV = V' - V = \alpha(dx + \frac{\partial y}{\partial x} \delta x) - \alpha dx = \alpha \frac{\partial y}{\partial x} \delta x$$

We have from the definition of Bulk modulus

$$K = -\frac{\Delta p}{\frac{dV}{V}} = -\frac{\Delta P}{\frac{\alpha \frac{\partial y}{\partial x} \delta x}{\alpha dx}}$$

Excess pressure on the layer of the medium (fluid)

$$\Delta P = -K \frac{\partial y}{\partial x}$$

This pressure is known as the sound pressure or acoustic pressure. Now, the excess pressure

$$= -\frac{\partial \Delta p}{\partial x} \delta x$$

This is negative because the unbalanced pressure is in the negative x-direction. So, the excess force acting on the layer

$$dF = -\alpha \frac{\partial \Delta p}{\partial x} \delta x = K \alpha \frac{\partial^2 y}{\partial x^2} \delta x$$

Again from Newton's second law we get

$$dF = \delta m \frac{\partial^2 y}{\partial t^2} \delta x = \alpha \rho \frac{\partial^2 y}{\partial t^2} \delta x$$

So, we get

$$\alpha \rho \frac{\partial^2 y}{\partial t^2} \delta x = K \alpha \frac{\partial^2 y}{\partial x^2} \delta x$$
$$\frac{\partial^2 y}{\partial t^2} = \frac{K}{\rho} \frac{\partial^2 y}{\partial x^2}$$

So, the wave velocity

$$v = \sqrt{\frac{K}{\rho}}$$

Q. Calculate the velocity of sound in a gas.

Newton's correction:

When a sound wave propagates in a gas, the pressure changes so rapidly that there is no change of temperature of the layer. Hence the process is isothermal. So, we can write

$$PV = \text{constant}$$

$$PdV + Vdp = 0$$

$$\frac{dP}{-\frac{dV}{V}} = P$$

$$K = P$$

So, velocity of sound

$$V_N = \sqrt{\frac{K}{\rho}} = \sqrt{\frac{P}{\rho}}$$

But $V_N < V_{\text{expt}}$, So there is a discrepancy between experiment and theory. This is removed by Laplace by considering adiabatic process.

$$PV^\gamma = \text{constant}$$

$$\gamma PV^{\gamma-1} dV + V^\gamma dP = 0$$

$$\gamma P dV = -V dp$$

$$\frac{dP}{-\frac{dV}{V}} = \gamma P$$

$$K = \gamma P$$

So, velocity of sound

$$V_L = \sqrt{\frac{K}{\rho}} = \sqrt{\frac{\gamma P}{\rho}}$$

Here, $V_L = V_{expt}$

Q. Calculate the velocity of longitudinal waves in a solid.

Let a longitudinal wave propagates along x-axis in a solid bar. Now, we consider a layer AB at a distance x from O and thickness of the layer AB = dx .

Let α is the area of the layer.

So, volume of the layer $V = \alpha dx$

Let there is an excess pressure between the faces of the layer AB. So the particles on the planes A and B are displaced due to the excess pressure ΔP produced by the progressive wave.

Excess compressive force $\Delta F = \Delta P \alpha$

Let displacement of the layer A is y and that of B is $y + dy$.

According to the fig.

$$OA' = x + y$$

$$OB' = x + dx + y + dy = x + dx + y + \frac{\partial y}{\partial x} dx$$

Thickness of the displaced layer

$$A'B' = OB' - OA' = (x + dx + y + \frac{\partial y}{\partial x} \delta x) - (x + y) = dx + \frac{\partial y}{\partial x} \delta x$$

Longitudinal strain

$$= A'B' - AB = dx + \frac{\partial y}{\partial x} \delta x - dx = \frac{\partial y}{\partial x} \delta x$$

We have from the definition of Young modulus

$$Y = -\frac{\frac{\Delta F}{\alpha}}{\frac{\frac{\partial y}{\partial x} \delta x}{\delta x}}$$

$$\Delta F = -Y\alpha \frac{\partial y}{\partial x}$$

This is negative because the unbalanced force is in the negative x-direction. So, the excess force acting on the layer

$$dF = -\alpha \frac{\partial \Delta F}{\partial x} \delta x = Y\alpha \frac{\partial^2 y}{\partial x^2} \delta x$$

Again from Newton's second law we get

$$dF = \delta m \frac{\partial^2 y}{\partial t^2} \delta x = \alpha \rho \frac{\partial^2 y}{\partial t^2} \delta x$$

So, we get

$$\alpha \rho \frac{\partial^2 y}{\partial t^2} \delta x = Y\alpha \frac{\partial^2 y}{\partial x^2} \delta x$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{Y}{\rho} \frac{\partial^2 y}{\partial x^2}$$

So, the wave velocity

$$v = \sqrt{\frac{Y}{\rho}}$$